

# Super Poisson-Lie symmetry of the $GL(1|1)$ WZNW model and worldsheet boundary conditions

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## Abstract

We show that the WZNW model on the Lie supergroup  $GL(1|1)$  has super Poisson-Lie symmetry with the dual Lie supergroup  $B \oplus A \oplus A_{1,1}.i$ . Then, we discuss about  $D$ -branes and worldsheet boundary conditions on supermanifolds, in general, and obtain the algebraic relations on the gluing supermatrix for the Lie supergroup case. Finally, using the supercanonical transformation description of the super Poisson-Lie T-duality transformation, we obtain formulae for the description of the dual gluing supermatrix, then, we find the gluing supermatrix for the WZNW model on  $GL(1|1)$  and its dual model. We also discuss about different boundary conditions.

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# 1 Introduction

Field theory and two-dimensional sigma models with supermanifolds as target space have recently received considerable attention, because of their relations to condensed matter physics and superstring models, respectively. As an example, the WZNW models on supergroups are related to local logarithmic conformal field theories [1, 2, 3]. The first attempt in this direction dates back to about 3 decades ago [4], where the flat space GS superstring action was reproduced as a WZNW type sigma model on the coset superspace  $(\frac{D=10 \text{ Poincare supergroup}}{SO(9,1)})$ . Then, this work is extended to the curved background [5] and shown that type IIB superstring on  $AdS_5 \times S^5$  can be constructed from sigma model on the coset superspace  $\frac{SU(2,2|4)}{SO(4,1) \times SO(5)}$ . After then, superstring theory on  $AdS_3 \times S^3$  is related to WZNW model on  $PSU(1,1|2)$  [6] and also superstring theory on  $AdS_2 \times S^2$  is related to sigma model on supercoset  $\frac{PSU(1,1|2)}{U(1) \times U(1)}$  [7]. There are also other works in this direction, see for instance [8].

On the other hand, T-duality is the most important symmetries of string theory [9]. Furthermore, Poisson-Lie T-duality, a generalization of T-duality, does not require existence of isometry in the original target manifold (as in usual T-duality) [10, 11]. So, the studies of Poisson-Lie T-duality in sigma models on supermanifolds and duality in superstring theories on AdS backgrounds are interesting problems. In the previous works [12] we extended Poisson-Lie symmetry to sigma models on supermanifolds and also constructed Poisson-Lie T-dual sigma models on Lie supergroups [13]. In this paper we show that the WZNW model on the Lie supergroup  $GL(1|1)$  has super Poisson-Lie symmetry with the dual Lie supergroup  $B \oplus A \oplus A_{1,1|.i}$ . In [14], it was shown that the mutually  $T$ -dual sigma models on the cosets  $D/G$  and  $D/\tilde{G}$  are the same, being equal to the WZNW model on  $R$ ; such that  $\mathcal{R}$  is the Lie subalgebra or  $R$  directly is identified with  $D/G$  and  $D/\tilde{G}$ . We note that until now there is only one example [15] for mutually  $T$ -dual sigma model on  $G$  or  $\tilde{G}$  such that one of the models is a WZNW model. In that example, the WZNW model is a constraint model because of singularity of the constant background matrix  $E(e)$ . Here, we first show that the WZNW model on the Lie supergroup  $GL(1|1)$  has super Poisson-Lie symmetry, then we obtain the mutually  $T$ -dual sigma models on  $GL(1|1)$  and its dual Lie supergroup  $B \oplus A \oplus A_{1,1|.i}$ . Note that the model on Lie supergroup  $B \oplus A \oplus A_{1,1|.i}$  is not a WZNW model, so it can not be conformal invariant. Furthermore, we discuss about  $D$ -branes and worldsheet boundary conditions on supermanifolds, in general, and specially on Lie supergroups. For the  $GL(1|1)$  Lie supergroup, the  $D$ -brane and worldsheet boundary conditions previously have been studied in [16] (see, also [17]). Here, we will present new boundary conditions for this Lie supergroup and will study the effect of super Poisson-Lie T-duality on these conditions.

The structure of this note is as follows. In Section 2 for self containing of the paper and introducing the notations, we review some aspects of the super Poisson-Lie T-dual sigma model on Lie supergroup [13]. In Section 3, by using of direct calculation of the super Poisson-Lie symmetry condition, we show that the WZNW model on  $GL(1|1)$  Lie supergroup has super Poisson-Lie symmetry when the dual Lie supergroup is  $B \oplus A \oplus A_{1,1|.i}$ . Then, we obtain the mutually T-dual sigma model on the Drinfel'd

superdouble supergroup ( $GL(1|1)$  ,  $B \oplus A \oplus A_{1,1|i}$ ) in Section 4; such that the original model is the WZNW model on the Lie supergroup  $GL(1|1)$ . In Section 5, we first discuss about  $D$ -branes and the worldsheet boundary conditions on supermanifolds and obtain general relation on the gluing supermatrix which defines the relation between left- and right-movers on the worldsheet boundary; then we present algebraic form of these conditions when the target space is a Lie supergroup. Then, using the supercanonical transformation description of the duality transformation, we derive a duality map for the gluing supermatrix which locally defines the properties of the  $D$ -brane. Finally, in the latter subsection of Section 5, we obtain the gluing supermatrices for the WZNW model on  $GL(1|1)$  and its dual model as six cases. Also, boundary conditions are discussed for the case 2.

## 2 Review of the super Poisson-Lie T-dual sigma models on supergroups

Let us start with a short review of the super Poisson-Lie symmetry [13] on supermanifolds<sup>1</sup>. In what follows we shall consider a nonlinear sigma model on a supermanifold  $M$  as<sup>2</sup>

$$S = -\frac{T}{2} \int d\tau d\sigma \left[ \sqrt{-h} h^{ab} \partial_a \Phi^\Upsilon \, {}_\Upsilon G_\Lambda(\Phi) \, \partial_b \Phi^\Lambda + \epsilon^{ab} \partial_a \Phi^\Upsilon \, {}_\Upsilon B_\Lambda(\Phi) \, \partial_b \Phi^\Lambda \right], \quad (2.1)$$

where  $h_{ab}$  and  $\epsilon^{ab}$  are the metric and antisymmetric tensor on the worldsheet, respectively, such that  $h \equiv \text{deth}_{ab}$  and the indices  $a, b = \tau, \sigma$ . The coordinates  $\Phi^\Upsilon$  include the bosonic coordinates  $X^\mu$  and the fermionic ones  $\Theta^\alpha$ , and the labels  $\Upsilon$  and  $\Lambda$  run over  $(\mu, \alpha)$ . The labels  $\mu$  and  $\alpha$  run from 0 to  $d_B - 1$  and from 1 to  $d_F$ , respectively, such that  $d_F$  is an even number<sup>3</sup>. We denote the dimension of the bosonic directions by  $d_B$  and the dimension of the fermionic directions by  $d_F$ . Thus, the superdimension of the supermanifold is written as  $(d_B|d_F)$ . One can write the action (2.1) in lightcone coordinates and then obtain

$$S = \frac{1}{2} \int d\xi^+ \wedge d\xi^- \, \partial_+ \Phi^\Upsilon \, {}_\Upsilon \mathcal{E}_\Lambda(\Phi) \, \partial_- \Phi^\Lambda, \quad (2.2)$$

where  $\partial_\pm$  are the derivatives with respect to the standard lightcone variables  $\xi^\pm \equiv \frac{1}{2}(\tau \pm \sigma)$  and  $\mathcal{E}_{\Upsilon\Lambda} = G_{\Upsilon\Lambda} + B_{\Upsilon\Lambda}$ .

Now we assume that the supergroup  $G$  acts freely on  $M$  from right. The Hodge star of Noether's current one-forms corresponding to the right action of the supergroup  $G$  on the target  $M$  of the sigma model (2.2) has the following form

$$\star_i J = (-1)^{\Upsilon+\Lambda} \, {}_i V^{(L,l)\Lambda} \, \partial_+ \Phi^\Upsilon \, {}_\Upsilon \mathcal{E}_{\Lambda\Upsilon} \, d\xi^+ - (-1)^\Lambda \, {}_i V^{(L,l)\Lambda} \, {}_\Lambda \mathcal{E}_{\Lambda\Upsilon} \, \partial_- \Phi^\Upsilon \, d\xi^-, \quad (2.3)$$

<sup>1</sup>Here we use the notation presented by DeWitt's in [18].

<sup>2</sup>Note that  $G_{\Upsilon\Lambda}$  and  $B_{\Upsilon\Lambda}$  are supersymmetric metric and antisupersymmetric tensor field, respectively, i.e.,

$$G_{\Upsilon\Lambda} = (-1)^{|\Upsilon||\Lambda|} G_{\Lambda\Upsilon}, \quad B_{\Upsilon\Lambda} = -(-1)^{|\Upsilon||\Lambda|} B_{\Lambda\Upsilon}.$$

We will assume that the metric  ${}_\Upsilon G_\Lambda$  is superinvertible and its superinverse is denoted  $G^{\Upsilon\Lambda}$ . In the above relations  $|\Upsilon|$  denotes the parity of  $\Upsilon$ , here and in the following we use the notation [18]  $(-1)^\Upsilon := (-1)^{|\Upsilon|}$ .

<sup>3</sup>For invertibility of the metric  $G_{\Upsilon\Lambda}$ ,  $d_F$  must be even.

where  ${}_iV^\Upsilon$ 's are the left invariant supervector fields (defined with left derivative)<sup>4</sup>. Now, we demand that the forms  $\star_i J$  on the extremal surfaces  $\Phi^\Upsilon(\xi^+, \xi^-)$  satisfy the Maurer-cartan equation [18]

$$d \star_i J = -(-1)^{jk} \frac{1}{2} \tilde{f}^{jk}_i \star_j J \wedge \star_k J, \quad (2.4)$$

where  $\tilde{f}^{jk}_i$  are structure constants of Lie superalgebra  $\tilde{\mathcal{G}}$  (the dual Lie superalgebra to  $\mathcal{G}$ ). Then, the condition of the super Poisson-Lie symmetry for the sigma model (2.2) is given by [18]

$$\mathcal{L}_{V_i}(\mathcal{E}_{\Upsilon\Lambda}) = (-1)^{i(\Upsilon+k)} \mathcal{E}_{\Upsilon\Xi} (V^{st})^\Xi_k (\tilde{\mathcal{Y}}_i)^{kj} V^\Omega_\Omega \mathcal{E}_\Lambda \quad (2.5)$$

where  $(\tilde{\mathcal{Y}}_i)^{jk} = -\tilde{f}^{jk}_i$  are the adjoint representations of Lie superalgebra  $\mathcal{G}$  and "st" stands for the *supertranspose* [18]. As mentioned in [13], the integrability condition for Lie superderivative gives compatibility between the structure constants of Lie superalgebras  $\mathcal{G}$  and  $\tilde{\mathcal{G}}$  which are well-known as the mixed super Jacobi identities of  $(\mathcal{G}, \tilde{\mathcal{G}})$  [19].

$$f^k_{ij} \tilde{f}^{ml}_k = (-1)^{il} f^m_{ik} \tilde{f}^{kl}_j + f^l_{ik} \tilde{f}^{mk}_j + f^m_{kj} \tilde{f}^{kl}_i + (-1)^{mj} f^l_{kj} \tilde{f}^{mk}_i. \quad (2.6)$$

In the following, we shall consider T-dual sigma model on supergroup  $G$ . To this end, suppose  $G$  acts transitively and freely on  $M$ ; then the target can be identified with the supergroup  $G$ . Now we assume  $\varepsilon^+$  is an  $(d_B|d_F)$ -dimensional linear subsuperspace and  $\varepsilon^-$  is its orthogonal complement such that  $\varepsilon^+ + \varepsilon^-$  span the Lie superalgebra  $\mathcal{D} = (\mathcal{G}|\tilde{\mathcal{G}})$ , i.e., Drinfel'd superdouble<sup>5</sup>. To determine a dual pair of the sigma models with the targets  $G$  and  $\tilde{G}$ , one can consider the following equation of motion for the mapping  $l(\xi^+, \xi^-)$  from the worldsheet into the Drinfel'd superdouble supergroup  $D$  [13]

$$\langle \partial_\pm l l^{-1}, \varepsilon^\mp \rangle = 0, \quad (2.7)$$

where  $\langle \cdot, \cdot \rangle$  means the invariant bilinear form on the superdouble. Using the Eq. (2.7) and the decomposition of an arbitrary element of  $D$  in the vicinity of the unit element of  $D$  as

$$l(\xi^+, \xi^-) = g(\xi^+, \xi^-) \tilde{h}(\xi^+, \xi^-), \quad g \in G, \quad \tilde{h} \in \tilde{G}, \quad (2.8)$$

we obtain

$$\langle g^{-1} \partial_\pm g + \partial_\pm \tilde{h} \tilde{h}^{-1}, g^{-1} \varepsilon^\mp g \rangle = 0, \quad (2.9)$$

for which

$$g^{-1} \varepsilon^\pm g = \text{Span}\{X_i \pm E_{ij}^\pm(g) \tilde{X}^j\}, \quad (2.10)$$

such that  $E_{ij}^- = (E_{ij}^+)^{st} = (-1)^{ij} E_{ji}^+$ ;  $X_i$  and  $\tilde{X}^i$  are basis of the respective Lie superalgebras  $\mathcal{G}$  and  $\tilde{\mathcal{G}}$ . It is crucial for super Poisson-Lie T-duality that the superalgebras

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<sup>4</sup>From now on we will omit the superscripts  $(L, l)$  on  ${}_iV^{(L, l)}$ .

<sup>5</sup>A Drinfel'd superdouble [20] is a Lie superalgebra  $\mathcal{D}$  which decomposes into the direct sum, as supervector spaces, of two maximally superisotropic Lie subsuperalgebras  $\mathcal{G}$  and  $\tilde{\mathcal{G}}$ , each corresponding to a Poisson-Lie supergroup ( $G$  and  $\tilde{G}$ ), such that the subsuperalgebras are duals of each other in the usual sense, i.e.,  $\tilde{\mathcal{G}} = \mathcal{G}^*$ .

generated by  $X_i$  and  $\tilde{X}^i$  form a pair of maximally superisotropic subsuperalgebras into the Drinfel'd superdouble so that

$$\begin{aligned} \langle X_i, X_j \rangle &= \langle \tilde{X}^i, \tilde{X}^j \rangle = 0, \\ \langle X_i, \tilde{X}^j \rangle &= (-1)^{ij} \langle \tilde{X}^j, X_i \rangle = (-1)^i \delta^j_i. \end{aligned} \quad (2.11)$$

Now, one can write the action (2.2) in the following form

$$S = \frac{1}{2} \int d\xi^+ \wedge d\xi^- R_+^{(l)i} F_j^+ \cdot jR_-^{(l)}, \quad (2.12)$$

where  $R_\pm^{(l)i}$ 's are right invariant one-forms with left derivative, i.e.,

$$R_+^{(l)i} = \partial_+ \Phi^r \cdot R^{(l)i} = (\partial_+ g g^{-1})^i, \quad (2.13)$$

$$jR_-^{(l)} = (R_-^{(l)st})^j \cdot \partial_- \Phi^r = (\partial_- g g^{-1})^j, \quad (2.14)$$

and<sup>6</sup>

$$F^+(g) = \left( \Pi(g) + (E^+)^{-1}(e) \right)^{-1}, \quad (2.15)$$

such that

$$\Pi^{ij}(g) = b^{ik}(g) \cdot k(a^{-1})^j(g), \quad (2.16)$$

where the matrices  $a(g)$  and  $b(g)$  are constructed using

$$g^{-1} X_i g = (-1)^j a_i^j(g) X_j, \quad (2.17)$$

$$g^{-1} \tilde{X}^i g = (-1)^j b^{ij}(g) X_j + d_j^i(g) \tilde{X}^j, \quad (2.18)$$

and consistency restricts them to obey

$$a(g^{-1}) = a^{-1}(g) = d^{st}(g), \quad \Pi(g) = -\Pi^{st}(g). \quad (2.19)$$

We expect that there exists an equivalent T-dual sigma model in which the roles of  $\mathcal{G}$  and  $\tilde{\mathcal{G}}$  are exchanged. So, one can repeat all steps of the previous construction to end up with the following T-dual sigma model

$$\tilde{S} = \frac{1}{2} \int d\xi^+ \wedge d\xi^- \tilde{R}_+^{(l)i} \tilde{F}^{+ij} \cdot j\tilde{R}_-^{(l)}, \quad (2.20)$$

where

$$\tilde{F}^+(\tilde{g}) = \left( \tilde{\Pi}(\tilde{g}) + (\tilde{E}^+)^{-1}(\tilde{e}) \right)^{-1}. \quad (2.21)$$

Indeed, at the origin of the supergroup ( $g = e$  and  $\tilde{g} = \tilde{e}$ ) the relation between the matrices  $E^\pm(e)$  and  $\tilde{E}^\pm(\tilde{e})$  are given by

$$E^\pm(e) \tilde{E}^\pm(\tilde{e}) = \tilde{E}^\pm(\tilde{e}) E^\pm(e) = I. \quad (2.22)$$

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<sup>6</sup>Here, one must use of superdeterminant and superinverse formulae [18].

### 3 Super Poisson-Lie symmetry of the $GL(1|1)$ WZNW model

The WZNW model based on supergroup  $G$  takes the following standard form

$$S_{WZNW}(g) = \frac{k}{4\pi} \int_{\Sigma} d\xi^+ \wedge d\xi^- < g^{-1} \partial_+ g, g^{-1} \partial_- g > \\ + \frac{k}{24\pi} \int_B < g^{-1} dg, [g^{-1} dg, g^{-1} dg] >, \quad (3.1)$$

where the integrations are over worldsheet  $\Sigma$  and a 3-dimensional manifold with boundary  $\partial B = \Sigma$ , respectively, and  $g^{-1} \partial_{\alpha} g$  are the left invariant one-forms (with left derivative) on supergroup  $G$  so that they may be expressed as

$$L_{\alpha}^{(l)} \equiv g^{-1} \partial_{\alpha} g = (-1)^i (g^{-1} \partial_{\alpha} g)^i X_i. \quad (3.2)$$

The WZNW action (3.1) then can be rewritten in terms of the  $L_{\alpha}^{(l)i}$ 's

$$S_{WZNW}(g) = \frac{k}{4\pi} \int_{\Sigma} d^2 \xi L_+^{(l)i} \Omega_j^{(l)} - \frac{k}{24\pi} \int_B d^3 \xi (-1)^{jk} \varepsilon^{\gamma\alpha\beta} L_{\gamma}^{(l)i} \Omega_l^{(l)} L_{\alpha}^{(l)j} (\mathcal{Y}^l)_{jk} L_{\beta}^{(l)k}, \quad (3.3)$$

where  $(\mathcal{Y}^l)_{jk} = -f_{jk}^l$  are the adjoint representations of Lie superalgebra  $\mathcal{G}$  and  $\Omega_{ij} = < X_i, X_j > = (-1)^{ij} \Omega_{ji}$  is non-degenerate supersymmetric ad-invariant metric on  $\mathcal{G}$ . Using the definition of metric  $\Omega_{ij}$  and ad-invariant inner product on  $\mathcal{G}$  as

$$< X_i, [X_j, X_k] > = < [X_i, X_j], X_k >, \quad (3.4)$$

we find

$$\mathcal{X}_i \Omega + (\mathcal{X}_i \Omega)^{st} = 0, \quad (3.5)$$

where  $(\mathcal{X}_i)_j^k = -f_{ij}^k$ . Before proceeding to write (3.1) on the Lie supergroup  $GL(1|1)$ , let us introduce the  $gl(1|1)$  Lie superalgebra. The Lie superalgebra  $gl(1|1)$  has  $(2|2)$ -superdimension with bosonic and fermionic generators denoted by  $H, Z$ <sup>7</sup> and by  $Q_+, Q_-$ , respectively. These four generators obey the following set of non-trivial (anti)commutation relations [25], [26], [1]

$$[H, Q_+] = Q_+, \quad [H, Q_-] = -Q_-, \quad \{Q_+, Q_-\} = Z. \quad (3.6)$$

Here, we obtain a non-degenerate general solution to Eq. (3.5) as follows

$$\Omega_{ij} = \begin{pmatrix} b & a & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & a \\ 0 & 0 & -a & 0 \end{pmatrix}, \quad a \in \mathfrak{R} - \{0\}, \quad b \in \mathfrak{R}. \quad (3.7)$$

In order to write (3.1) explicitly, we need to find the  $L_{\alpha}^{(l)}$ 's. To this purpose we use the following parametrization of the Lie supergroup  $GL(1|1)$  [1]:

$$g = e^{\chi Q_-} e^{yH+xZ} e^{\psi Q_+}. \quad (3.8)$$

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<sup>7</sup> $Z$  is central generator, i.e., it commutes with all other elements of  $gl(1|1)$ .

The fields  $x(\tau, \sigma)$  and  $y(\tau, \sigma)$  are bosonic while  $\psi(\tau, \sigma)$  and  $\chi(\tau, \sigma)$  are fermionic. Inserting our specific choice of the parametrization (3.8), the  $L_\alpha^{(l)}$ 's take the following form

$$L_\alpha^{(l)} = \partial_\alpha y H + \partial_\alpha x Z - \partial_\alpha \chi e^y \psi Z + \partial_\alpha y \psi Q_+ + \partial_\alpha \chi e^y Q_-, \quad (3.9)$$

for which we can read off the  $L_\alpha^{(l)i}$  and the terms that are being integrated over in (3.3) are calculated to be

$$L_+^{(l)i} {}_i\Omega_j L_-^{(l)j} = a[\partial_+ y \partial_- x + \partial_+ x \partial_- y - \partial_+ \psi e^y \partial_- \chi + \partial_+ \chi e^y \partial_- \psi], \quad (3.10)$$

$$\begin{aligned} (-1)^{jk} L_\gamma^{(l)i} {}_i\Omega_l L_\alpha^{(l)j} (\mathcal{Y}^l)_{jk} L_\beta^{(l)k} &= -a \partial_\gamma [-\partial_\alpha \psi e^y \partial_\beta \chi - \partial_\alpha \chi e^y \partial_\beta \psi \\ &\quad + \partial_\alpha e^y \psi \partial_\beta \chi - \partial_\beta e^y \psi \partial_\alpha \chi + \partial_\alpha e^y \chi \partial_\beta \psi - \partial_\beta e^y \chi \partial_\alpha \psi]. \end{aligned} \quad (3.11)$$

Finally, the  $GL(1|1)$  WZNW action looks like

$$S_{WZNW}(g) = \frac{ak}{4\pi} \int_\Sigma d\xi^+ \wedge d\xi^- (\partial_+ y \partial_- x + \partial_+ x \partial_- y - 2\partial_+ \psi e^y \partial_- \chi). \quad (3.12)$$

Here, we have assumed that  $b = 0$  in (3.7). One can derive the  $GL(1|1)$  WZNW action deduced in [1, 17] by choosing  $a = -1$ . On the other hand, by rescaling  $a$  to  $-\frac{2\pi}{k}$  and using integrating by parts, the action (3.12) is reduced to

$$\begin{aligned} S_{WZNW}(g) &= \frac{1}{2} \int d\xi^+ \wedge d\xi^- (-\partial_+ y \partial_- x - \partial_+ x \partial_- y + \partial_+ \psi e^y \partial_- \chi \\ &\quad - \partial_+ y e^y \psi \partial_- \chi - \partial_+ \chi e^y \psi \partial_- y - \partial_+ \chi e^y \partial_- \psi). \end{aligned} \quad (3.13)$$

By regarding this action as a sigma model action of the form (2.2), we can read off the background matrix as follows:

$$\mathcal{E}_{\Upsilon\Lambda} = \begin{pmatrix} 0 & -1 & 0 & -\psi e^y \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -e^y \\ \psi e^y & 0 & e^y & 0 \end{pmatrix}. \quad (3.14)$$

In the following, we shall investigate that the  $GL(1|1)$  WZNW model has super Poisson-Lie symmetry. To this end, we need the left invariant supervector fields (with left derivative) on the  $GL(1|1)$ . Substituting  $L^{(l)i} = \vec{d} \Phi^\Upsilon L^{(l)i}$  and  ${}_iV = {}_iV^\Upsilon \frac{\vec{\partial}}{\partial \Phi^\Upsilon}$  into  $\langle {}_iV, L^{(l)j} \rangle = {}_i\delta^j$  we then obtain [13]

$${}_iV^\Upsilon = (\Upsilon L^{(l)i})^{-1}. \quad (3.15)$$

Thus, using the Eqs. (3.9) and (3.15),  ${}_iV$ 's take the following form

$$\begin{aligned} {}_H V &= \frac{\vec{\partial}}{\partial y} - \psi \frac{\vec{\partial}}{\partial \psi}, \\ {}_Z V &= \frac{\vec{\partial}}{\partial x}, \end{aligned}$$

$$\begin{aligned}
{}_{q_+}V &= -\frac{\vec{\partial}}{\partial\psi}, \\
{}_{q_-}V &= -\psi\frac{\vec{\partial}}{\partial x} - e^{-y}\frac{\vec{\partial}}{\partial\chi}.
\end{aligned} \tag{3.16}$$

Now, putting the relations (3.14) and (3.16) on the right hand side of Eq. (2.5) and by direct calculation of the Lie superderivatie corresponding to the  ${}_iV$  of  $\mathcal{E}_{\text{r}\Lambda}$  [13], then one can find the non-zero structure constants of the dual pair to the  $gl(1|1)$  Lie superalgebra as

$$\tilde{f}^{23}_3 = -\tilde{f}^{32}_3 = 1. \tag{3.17}$$

In our recent work [26], we classified all the dual Lie superalgebras to the  $gl(1|1)$ . In [26] the dual Lie superalgebra (3.17) has been labeled to the  $\mathcal{B} \oplus \mathcal{A} \oplus \mathcal{A}_{1,1|.i}$ , such that this Lie superalgebra is a decomposable Lie superalgebra of the type (2|2) where is generated by the set of bosonic generators  $\tilde{X}^1 = \tilde{H}$ ,  $\tilde{X}^2 = \tilde{Z}$  and fermionic ones  $\tilde{X}^3 = \tilde{Q}_+$ ,  $\tilde{X}^4 = \tilde{Q}_-$  with the commutation relations of (3.17). In the next section, we will show that the original T-dual sigma model on the Drinfel'd superdouble supergroup  $(GL(1|1), B \oplus A \oplus A_{1,1|.i})$  is equivalent to the  $GL(1|1)$  WZNW model.

## 4 Super Poisson-Lie dualizable sigma model on the $(GL(1|1), B \oplus A \oplus A_{1,1|.i})$

In this section we will first introduce the Drinfel'd superdouble generated by the  $gl(1|1)$  Lie superalgebra and it's dual  $\tilde{\mathcal{G}} = \mathcal{B} \oplus \mathcal{A} \oplus \mathcal{A}_{1,1|.i}$ . We construct in particular, super Poisson-Lie dualizable sigma models on the  $GL(1|1)$  and  $B \oplus A \oplus A_{1,1|.i}$ . The Manin supertriple  $(gl(1|1), \mathcal{B} \oplus \mathcal{A} \oplus \mathcal{A}_{1,1|.i})$  possesses four bosonic generators and four fermionic ones. We shall denote the bosonic generators by  $\{H, Z, \tilde{H}, \tilde{Z}\}$  and use  $\{Q_+, Q_-, \tilde{Q}_+, \tilde{Q}_-\}$  for fermionic generators. The relations between these elements are given by [26]

$$\begin{aligned}
[H, Q_+] &= Q_+, & [H, Q_-] &= -Q_-, & \{Q_+, Q_-\} &= Z, \\
[\tilde{Z}, \tilde{Q}_+] &= \tilde{Q}_+, & [H, \tilde{Q}_+] &= -\tilde{Q}_+, & [H, \tilde{Q}_-] &= \tilde{Q}_-, \\
[\tilde{Z}, Q_-] &= \tilde{Q}_+, & \{Q_-, \tilde{Q}_-\} &= \tilde{H}, & \{Q_+, \tilde{Q}_+\} &= Z - \tilde{H}, \\
[\tilde{Z}, Q_+] &= -Q_+ + \tilde{Q}_-.
\end{aligned} \tag{4.1}$$

In addition, the elements  $Z$  and  $\tilde{H}$  are central.

### 4.1 The original model

There exist various choices that come with different parametrizations of the Lie supergroup  $GL(1|1)$ . The convenient parametrization for us is the same of (3.8). Using the parametrization (3.8) and the relation (2.13) we have explicitly

$$R_{\pm}^{(l)H} = \partial_{\pm}y,$$



$$\begin{aligned}
R_{\pm}^{(l)Z} &= \partial_{\pm}x + \partial_{\pm}\psi e^y\chi, \\
R_{\pm}^{(l)Q_+} &= -e^y\partial_{\pm}\psi, \\
R_{\pm}^{(l)Q_-} &= -\chi\partial_{\pm}y - \partial_{\pm}\chi.
\end{aligned} \tag{4.2}$$

By a direct application of formulae (2.17) and (2.18) the super Poisson structure is work out as follows:

$$\Pi^{ij}(g) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \psi e^y & 0 \\ 0 & -\psi e^y & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.3}$$

Then, choosing the inverse sigma model matrix  $(E^+)^{-1}(e)$  at the unit element of  $GL(1|1)$  as

$$(E^{+-1})^{ij}(e) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{4.4}$$

and finally, using the relations (2.15) and (4.2), the original model action (2.12) is obtained to be of the form

$$\begin{aligned}
S = \frac{1}{2} \int d\xi^+ \wedge d\xi^- & (-\partial_+y\partial_-x - \partial_+x\partial_-y + \partial_+\psi e^y\partial_-x \\
& - \partial_+y e^y\psi\partial_-x - \partial_+\chi e^y\psi\partial_-y - \partial_+\chi e^y\partial_-y).
\end{aligned} \tag{4.5}$$

By identifying the above action with the sigma model of the form (2.2), one can read off the background supersymmetric metric  $G_{\Upsilon\Lambda}$  and antisupersymmetric tensor field  $B_{\Upsilon\Lambda}$  as follows:

$$G_{\Upsilon\Lambda} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -e^y \\ 0 & 0 & e^y & 0 \end{pmatrix}, \quad B_{\Upsilon\Lambda} = \begin{pmatrix} 0 & 0 & 0 & -\psi e^y \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \psi e^y & 0 & 0 & 0 \end{pmatrix}. \tag{4.6}$$

This result is identical to the conclusion of (3.13). Thus, we showed that the original T-dual sigma model on the Drinfel'd superdouble supergroup  $(GL(1|1), B \oplus A \oplus A_{1,1}|i)$  is equivalent to the  $GL(1|1)$  WZNW model.

## 4.2 The dual model

In the same way to construct the dual model on the Lie supergroup  $B \oplus A \oplus A_{1,1}|i$ , we use of the following parametrization

$$\tilde{g} = e^{\tilde{x}\tilde{Q}_-} e^{\tilde{y}\tilde{H} + \tilde{x}\tilde{Z}} e^{\tilde{\psi}\tilde{Q}_+}. \tag{4.7}$$

By using the (4.7) we find

$$\partial_+\tilde{g}\tilde{g}^{-1} = \partial_+\tilde{y}\tilde{H} + \partial_+\tilde{x}\tilde{Z} + \partial_+\tilde{\psi}e^{\tilde{x}}\tilde{Q}_+ + \partial_+\tilde{\chi}\tilde{Q}_-, \tag{4.8}$$

from which we can read off the  $\tilde{R}_{+i}^{(l)}$ 's and compute

$$\tilde{\Pi}_{ij}(\tilde{g}) = \begin{pmatrix} 0 & 0 & -e^{\tilde{x}}\tilde{\psi} & \tilde{\chi} \\ 0 & 0 & 0 & 0 \\ e^{\tilde{x}}\tilde{\psi} & 0 & 0 & 1 - e^{\tilde{x}} \\ -\tilde{\chi} & 0 & 1 - e^{\tilde{x}} & 0 \end{pmatrix}. \quad (4.9)$$

Finally, using the Eq. (2.21) and the condition (2.22), the dual model action will have the following form

$$\begin{aligned} \tilde{S} = & \frac{1}{2} \int d\xi^+ \wedge d\xi^- \left[ -\partial_+ \tilde{y} \partial_- \tilde{x} - \partial_+ \tilde{x} \partial_- \tilde{y} + \frac{2}{e^{\tilde{x}} - 2} (-\partial_+ \tilde{x} \tilde{\psi} \tilde{\chi} \partial_- \tilde{x} \right. \\ & \left. + \partial_+ \tilde{x} \tilde{\psi} \partial_- \tilde{\chi} + \partial_+ \tilde{\psi} \tilde{\chi} (e^{\tilde{x}} - 1) \partial_- \tilde{x} - \partial_+ \tilde{\psi} \partial_- \tilde{\chi}) - 2\partial_+ \tilde{\chi} \tilde{\psi} \partial_- \tilde{x} \right]. \end{aligned} \quad (4.10)$$

## 5 *D*-branes on supermanifolds and worldsheet boundary conditions

To study *D*-branes on supermanifolds, we impose the boundary conditions for the bosonic and fermionic coordinates. Therefore, *D*-branes on supermanifolds have more formations than those on manifolds. For investigating *D*-branes, we must study the boundary conditions on the worldsheet. The open string may either move about freely, in which case its ends obey Neumann boundary conditions; or the ends of the string may be confined to a subsuperspace, corresponding to Dirichlet conditions. One must impose the Dirichlet or the Neumann conditions for each bosonic on the boundary. Note that for supermanifolds, since the two fields  $\Theta^\alpha$  and  $\Theta^{\alpha+1}$  become evident in pairs in the action [21], so one must more careful study the fermionic parts  $\Theta^\alpha$ . Furthermore, since the following conditions must be satisfied on the boundary

$$\delta\Theta^\alpha \partial_\sigma \Theta^{\alpha+1} = 0, \quad \delta\Theta^{\alpha+1} \partial_\sigma \Theta^\alpha = 0, \quad (5.1)$$

so, we need to impose the same boundary conditions for each pair of fermionic directions. Thus, the boundary conditions require that  $\delta\Theta^\alpha = \delta\Theta^{\alpha+1} = 0$  or  $\partial_\sigma \Theta^\alpha = \partial_\sigma \Theta^{\alpha+1} = 0$ . If the numbers of the directions with the Neumann conditions be presented by  $p+1$  for the bosonic directions and  $r$  for the fermionic directions, then,  $r$  must be an even number. *D*-branes with these configurations are called *Dp|r*-branes [21]. Consider a  $(d_B|d_F)$ -dimensional target space with *Dp|r*-branes, i.e., there are  $d_B - (p+1)$  Dirichlet directions along which the field  $X^\mu$  is frozen ( $\partial_0 X^a = 0, a = p+1, \dots, d_B - 1$ ). At any given point on a *Dp*-brane we can choose local coordinates such that  $X^a$  are the directions normal to the brane and  $X^m$  ( $m = 0, \dots, p$ ) are coordinates on the brane. But, for the fermionic part, we have  $d_F - r$  Dirichlet directions, where  $d_F - r$  is an even number.

### 5.1 *Worldsheet boundary conditions*

The worldsheet boundary is by definition confined to a *D*-brane. Since the boundary relates left-moving fields  $\partial_+ \Phi^r$  to the right-moving fields  $\partial_- \Phi^r$ , we make a general ansatz

for this relation. The goal is then to find the restrictions on this ansatz arising from varying the action (2.1). The most general local boundary condition may be expressed as<sup>8</sup>

$$\partial_- \Phi^{\mathbf{r}} = \mathcal{R}^{\mathbf{r}}_{\Lambda}(\Phi) \partial_+ \Phi^{\Lambda}, \quad (5.2)$$

where  $\mathcal{R}^{\mathbf{r}}_{\Lambda}$  is a locally defined object which is called the *gluing supermatrix*. Now we assume that  $\mathcal{R}^{\mathbf{r}}_{\Lambda}$  is in the form of a  $2 \times 2$  block matrix as

$$\mathcal{R}^{\mathbf{r}}_{\Lambda}(\Phi) = \left( \begin{array}{c|c} \mathcal{R}^{\mu}_{\nu} & \mathcal{R}^{\mu}_{\beta} \\ \hline \mathcal{R}^{\alpha}_{\nu} & \mathcal{R}^{\alpha}_{\beta} \end{array} \right), \quad (5.3)$$

where the elements of the submatrices  $\mathcal{R}^{\mu}_{\nu}$  and  $\mathcal{R}^{\alpha}_{\beta}$  are  $c$ -numbers, while the elements of the submatrices  $\mathcal{R}^{\mu}_{\beta}$  and  $\mathcal{R}^{\alpha}_{\nu}$  are  $a$ -numbers.

These boundary conditions have to preserve conformal invariance at the boundary. We know that each symmetry corresponds to a conserved current, obtained by varying the action with respect to the appropriate field. In the case of conformal invariance, the corresponding current is the stress energy-momentum tensor and is derived by varying the action (2.1) with respect to the metric  $h_{ab}$ . Its components in lightcone coordinates are

$$T_{\pm\pm} = (-1)^{\mathbf{r}} \partial_{\pm} \Phi^{\mathbf{r}} G_{\mathbf{r}\Lambda}(\Phi) \partial_{\pm} \Phi^{\Lambda}. \quad (5.4)$$

The  $T_{++}$  component depends only on  $\xi^+$ , and is called the left-moving current, whereas  $T_{--}$  depends only on  $\xi^-$  and is referred to as right-moving current. To ensure conformal symmetry on the boundary, we need to impose boundary conditions on the currents (5.4). In general, we find the boundary condition for a given current by using its associated charge. Applied to the stress tensor, the result is

$$T_{++} - T_{--} = 0. \quad (5.5)$$

Now, using the Eqs. (5.2), (5.4) and (5.5) we find

$$(-1)^{\Omega} (\mathcal{R}^{st})_{\mathbf{r}}^{\Omega} G_{\Omega\Xi} \mathcal{R}^{\Xi}_{\Lambda} = G_{\mathbf{r}\Lambda}. \quad (5.6)$$

Thus, in this way, we have derived the condition for conformal invariance on the boundary in a sigma model on supermanifold. In the following, we define a Dirichlet projector  $\mathcal{Q}^{\mathbf{r}}_{\Lambda}$  on the worldsheet boundary, which projects vectors onto the space normal to the brane. These vectors (Dirichlet vectors) are eigenvectors of  $\mathcal{R}^{\mu}_{\nu}(\mathcal{R}^{\alpha}_{\beta})$  with eigenvalue  $-1$ . Hence  $\mathcal{Q}^{\mathbf{r}}_{\Lambda}$  is given by the following axioms

$$\begin{aligned} \mathcal{Q}^2 &:= \mathcal{Q}^{\mathbf{r}}_{\Xi} \mathcal{Q}^{\Xi}_{\Lambda} = \mathcal{Q}^{\mathbf{r}}_{\Lambda}, \\ \mathcal{Q}^{\mathbf{r}}_{\Xi} \mathcal{R}^{\Xi}_{\Lambda} &= \mathcal{R}^{\mathbf{r}}_{\Xi} \mathcal{Q}^{\Xi}_{\Lambda} = -\mathcal{Q}^{\mathbf{r}}_{\Lambda}. \end{aligned} \quad (5.7)$$

Similarly, we may define a Neumann projector  $\mathcal{N}^{\mathbf{r}}_{\Lambda}$  which projects vectors onto the target space of the brane (vectors target to the brane are eigenvectors of  $\mathcal{R}^{\mu}_{\nu}(\mathcal{R}^{\alpha}_{\beta})$  with eigenvalue 1) and is defined as complementary to  $\mathcal{Q}^{\mathbf{r}}_{\Lambda}$ , i.e.,

$$\mathcal{N}^{\mathbf{r}}_{\Lambda} := \delta^{\mathbf{r}}_{\Lambda} - \mathcal{Q}^{\mathbf{r}}_{\Lambda}. \quad (5.8)$$

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<sup>8</sup>We note that the conditions (5.2), (5.7), (5.10) and (5.12) are a generalization of the bosonic conditions in [22].

In addition, by contracting (5.8) with  $\mathcal{Q}^\Lambda_\Xi$  and using (5.7), we then obtain

$$\mathcal{N}^\Upsilon_\Lambda \mathcal{Q}^\Lambda_\Xi = 0. \quad (5.9)$$

Also, the Neumann projector satisfy the following conditions

$$(-1)^\Omega (\mathcal{N}^{st})^\Omega_\Upsilon (\mathcal{E}^{st})_{\Omega\Xi} \mathcal{N}^\Xi_\Lambda - (-1)^\Xi (\mathcal{N}^{st})^\Xi_\Upsilon \mathcal{E}^\Xi_{\Xi\Omega} \mathcal{N}^\Omega_\Delta \mathcal{R}^\Delta_\Lambda = 0, \quad (5.10)$$

$$(-1)^\Omega (\mathcal{N}^{st})^\Omega_\Upsilon G_{\Omega\Xi} \mathcal{Q}^\Xi_\Lambda = 0, \quad (5.11)$$

note that for a spacefilling brane (when all directions are Neumann or  $\mathcal{Q}^\Upsilon_\Lambda = 0$ ) Eq. (5.10) implies that  $\mathcal{R}^\Upsilon_\Xi = (-1)^\Lambda (\mathcal{E}^{-1})^{\Upsilon\Lambda} (\mathcal{E}^{st})_{\Lambda\Xi}$ .

Before proceeding to discuss the dual conditions, let us write down the boundary conditions (5.2), (5.6), (5.7), (5.10) and (5.11) in the Lie superalgebra frame. These conditions read

$$R_-^{(l)i} = \mathcal{R}^i_j R_+^{(l)j}, \quad (5.12)$$

$$(-1)^k (\mathcal{R}^{st})^k_i \Omega_{kl} \mathcal{R}^l_j = \Omega_{ij}, \quad (5.13)$$

$$\mathcal{Q}^i_j \mathcal{R}^j_k = \mathcal{R}^i_j \mathcal{Q}^j_k = -\mathcal{Q}^i_k, \quad (5.14)$$

$$(-1)^k (\mathcal{N}^{st})^k_i (F^{+st})_{kl} \mathcal{N}^l_j - (-1)^l (\mathcal{N}^{st})^l_i F_{lk}^+ \mathcal{N}^k_m \mathcal{R}^m_j = 0, \quad (5.15)$$

$$(-1)^j (\mathcal{N}^{st})^j_i \Omega_{jk} \mathcal{Q}^k_l = 0, \quad (5.16)$$

where

$$\mathcal{R}^i_j = (R^{(l)st})^i_\Upsilon \mathcal{R}^\Upsilon_\Lambda (R^{(l)-st})^\Lambda_j, \quad \Omega_{ij} = (-1)^\Upsilon (R^{(l)-1})^\Upsilon_i G_{\Upsilon\Lambda} (R^{(l)-st})^\Lambda_j, \quad (5.17)$$

$$\mathcal{N}^i_j = (R^{(l)st})^i_\Upsilon \mathcal{N}^\Upsilon_\Lambda (R^{(l)-st})^\Lambda_j, \quad \mathcal{Q}^i_j = (R^{(l)st})^i_\Upsilon \mathcal{Q}^\Upsilon_\Lambda (R^{(l)-st})^\Lambda_j. \quad (5.18)$$

Note that the object  $\mathcal{R}^i_j$  is a gluing map between currents at the worldsheet boundary. Since it maps  $R_+^{(l)j}$  to  $R_-^{(l)i}$ , which are elements of the Lie superalgebra, it is clearly a map from the Lie superalgebra into itself. So, it may be assumed to be a constant Lie superalgebra automorphism, i.e., it preserves the Lie superalgebra structure.

## 5.2 Supercanonical transformations

In this subsection, by using the super Poisson-Lie T-duality transformation as a supercanonical transformation, we derive a duality map for the gluing supermatrix which locally defines the properties of the  $D$ -brane. In [23], Sfetsos formulated Poisson-Lie T-duality as an explicit transformation between the canonical variables of the two dual sigma models. Here, we generalize the classical canonical transformation on Lie group [23, 24] to the Lie supergroup. This transformation on the Lie supergroup  $G$  between the supercanonical pairs of variables  $(R_\sigma^{(l)i}, P_i)$  and  $((\tilde{R}_\sigma^{(l)})_j, \tilde{P}^j)$  is given by

$$R_\sigma^{(l)i} = \left( \delta^i_j - (-1)^k \Pi^{ik} \tilde{\Pi}_{kj} \right) \tilde{P}^j - (-1)^k \Pi^{ik} (\tilde{R}_\sigma^{(l)})_k, \quad (5.19)$$

$$P_i = \tilde{\Pi}_{ij} \tilde{P}^j + (\tilde{R}_\sigma^{(l)})_i, \quad (5.20)$$

where<sup>9</sup>

$$R_\sigma^{(l)i} = \frac{1}{2} \left( R_+^{(l)i} - R_-^{(l)i} \right), \quad (5.21)$$

$$\begin{aligned} P_i &= (-1)^\Upsilon (R^{(l)-1})_i{}^\Upsilon P_\Upsilon = (-1)^\Upsilon (R^{(l)-1})_i{}^\Upsilon \frac{L \overleftarrow{\delta}}{\delta(\partial_\tau \Phi^\Upsilon)} \\ &= \frac{1}{2} \left( (F^{+st})_{ij} R_+^{(l)j} + F_{ij}^+ R_-^{(l)j} \right), \end{aligned} \quad (5.22)$$

and similarly for the corresponding tilded symbols. To find the dual boundary conditions, one must find a transformation from  $R_\pm^{(l)}$  to  $\tilde{R}_\pm^{(l)}$ . For this purpose, we use Eqs. (2.15), (2.21), (5.21) and (5.22) to rewrite the supercanonical transformations (5.19) and (5.20) as follows:

$$(\tilde{R}_+^{(l)})_i = (-1)^l (\tilde{F}^{+-st})_{ij} (E^{+-st})^{jl} (e) (F^{+st})_{lk} R_+^{(l)k}, \quad (5.23)$$

$$(\tilde{R}_-^{(l)})_i = -(-1)^l (\tilde{F}^{+-1})_{ij} (E^{+-1})^{jl} (e) F_{lk}^+ R_-^{(l)k}. \quad (5.24)$$

Now, by using the above relations, the boundary condition (5.12) takes the following form<sup>10</sup>

$$(\tilde{R}_-^{(l)})_i = (-1)^j \tilde{\mathcal{R}}_i{}^j (\tilde{R}_+^{(l)})_j, \quad (5.25)$$

in which

$$\tilde{\mathcal{R}}_i{}^j = -(-1)^{l+p} (\tilde{F}^{+-1})_{ik} (E^{+-1})^{kl} (e) F_{lm}^+ \mathcal{R}_n^m (F^{+-st})^{np} (E^{+st})_{pq} (e) (\tilde{F}^{+st})^{qj}. \quad (5.26)$$

The above relation is the transformation of the gluing supermatrix. By using the (5.26) and the rules of supertranspose [18] we have  $s\det({}_i \tilde{\mathcal{R}}^j) = s\det(-\mathcal{R}^i{}_j)$ . This is a result that will be useful in the next subsection. Furthermore, again by use of the (5.26) one can determine the form of the dual Neumann and Dirichlet projectors  $\tilde{\mathcal{N}}$  and  $\tilde{\mathcal{Q}}$  via the definition  $(-1)^j \tilde{\mathcal{R}}_i{}^j \tilde{\mathcal{Q}}_j{}^k = (-1)^j \tilde{\mathcal{Q}}_i{}^j \tilde{\mathcal{R}}_j{}^k = -\tilde{\mathcal{Q}}_i{}^k$  and so on.

Similarly, to obtain the transformation of the metric on Lie superalgebra we use the relation (5.13). Thus, the dual of the (5.13) is found to be

$$(-1)^l (\tilde{\mathcal{R}}^{st})^i{}_k \tilde{\Omega}^{kl} \tilde{\mathcal{R}}_l{}^j = \tilde{\Omega}^{ij}, \quad (5.27)$$

where

$$\tilde{\Omega}^{ij} = (-1)^{l+n+p} (\tilde{F}^+)^{il} E_{lm}^+ (e) (F^{+-1})^{mn} \Omega_{nk} (F^{+-st})^{kp} (E^{+st})_{pq} (e) (\tilde{F}^{+st})^{qj}. \quad (5.28)$$

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<sup>9</sup>For calculating  $P_\Upsilon$  in (5.22), we use the Lagrangian (2.12).

<sup>10</sup>Here in the Lie superalgebra frame,  $\tilde{\mathcal{R}}^\Upsilon{}_\Lambda$  reads

$$\tilde{\mathcal{R}}_i{}^j = (\tilde{R}^{(l)st})_{i\Upsilon} \tilde{\mathcal{R}}^\Upsilon{}_\Lambda (\tilde{R}^{(l)-st})^{\Lambda j}.$$

### 5.3 Example

We now investigate the consequences of the duality transformation of the gluing supermatrix for the Drinfel'd superdouble  $(gl(1|1), \mathcal{B} \oplus \mathcal{A} \oplus \mathcal{A}_{1,1}|i)$ . The (anti)commutation relations of this superdouble and the super Poisson-Lie T-dual sigma models have been explicitly worked out in section 4. The constant background at the identity as the relation (4.4) and the super Poisson brackets have been given by the relations (4.3) and (4.9). Thus, using the relations (2.15), (2.21) and (2.22), the background fields  $F_{ij}^+(g)$  and  $\tilde{F}^{+ij}(\tilde{g})$  read

$$F_{ij}^+(g) = \begin{pmatrix} 0 & -1 & 0 & e^y \psi \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ -e^y \psi & 0 & 1 & 0 \end{pmatrix}, \quad (5.29)$$

$$\tilde{F}^{+ij}(\tilde{g}) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & \frac{-2\tilde{\psi}\tilde{\chi}}{e^{\tilde{x}}-2} & \tilde{\chi}e^{-\tilde{x}} & \frac{-\tilde{\psi}e^{\tilde{x}}}{e^{\tilde{x}}-2} \\ 0 & \frac{\tilde{\chi}}{e^{\tilde{x}}-2} & 0 & \frac{1}{e^{\tilde{x}}-2} \\ 0 & -\tilde{\psi} & e^{-\tilde{x}} & 0 \end{pmatrix}. \quad (5.30)$$

In this example (a supergroup with (2|2)-dimension), we have the following six different types of  $D$ -branes where for each of these cases we find the dual gluing supermatrix.

**Case 1:** Case 1 refers to  $D_{(-1)|0}$ -brane. The corresponding gluing supermatrix is given by

$$\mathcal{R}_j^i = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (5.31)$$

This choice means that all directions are Dirichlet, i.e.,  $\mathcal{Q}_j^i = \delta_j^i$  and  $\mathcal{N}_j^i = 0$ . Then, using the relation (4.4) and by substituting (5.29) and (5.30) into (5.26), the dual gluing supermatrix reads

$${}_i\tilde{\mathcal{R}}^j = \begin{pmatrix} 1 & \tilde{A}_1 & -\frac{2\tilde{\chi}+2\psi e^y}{e^{\tilde{x}}-2} & 2\tilde{\psi} \\ 0 & 1 & 0 & 0 \\ 0 & \frac{-2\tilde{\psi}e^{\tilde{x}}}{e^{\tilde{x}}-2} & \frac{-e^{\tilde{x}}}{e^{\tilde{x}}-2} & 0 \\ 0 & \tilde{A}_2 & 0 & 2e^{-\tilde{x}} - 1 \end{pmatrix}, \quad (5.32)$$

where

$$\tilde{A}_1 = \frac{2}{e^{\tilde{x}}-2} [\psi\tilde{\psi}e^{\tilde{x}+y}(e^{\tilde{x}}-3) + 2\tilde{\psi}\tilde{\chi}], \quad \tilde{A}_2 = 2\psi e^y(e^{\tilde{x}}-2) - 2\tilde{\chi}e^{-\tilde{x}}.$$

The superdeterminant is  $sdet({}_i\tilde{\mathcal{R}}^j) = sdet(-\mathcal{R}_j^i) = 1$ , so the dual brane may include the following directions, for some special backgrounds

- (i)  $D_{(-1)|0}$ -brane, with the Neumann projector  ${}_i\tilde{\mathcal{N}}^j = 0$ ,
- (ii)  $D_{(-1)|2}$ -brane, with  ${}_i\tilde{\mathcal{N}}^j = \text{diag}(0, 0, 1, 1)$ ,
- (iii)  $D_{1|0}$ -brane, with  ${}_i\tilde{\mathcal{N}}^j = \text{diag}(1, 1, 0, 0)$ ,

(iv)  $D_{1|2}$ -brane (spacefilling), with  ${}_i\tilde{\mathcal{N}}^j = {}_i\delta^j$ .

For the subcase (i), the only solution is  ${}_i\tilde{\mathcal{R}}^j = -{}_i\delta^j$ , i.e., all directions are Dirichlet, such that Eq. (5.26) reduces to  $(-1)^n \tilde{F}^{+in} (\tilde{F}^{+-st})_{nj} = -(-1)^{k+m} (E^{+-1})^{ik} F_{kl}^+ (F^{+-st})^{lm} (E^{+st})_{mj}$ . For the subcase (ii), the dual brane has two Dirichlet directions for the bosonic part and two Neumann directions for the fermionic one. In contrast to the subcase (ii), in the subcase (iii), the dual brane has zero Dirichlet directions for the bosonic part and two Dirichlet directions for the fermionic one. In the latter subcase, the only solution is  ${}_i\tilde{\mathcal{R}}^j = {}_i\delta^j$ , i.e., the dual brane has zero Dirichlet directions and the relation (5.26) reduces to  $\delta^i_j = (-1)^{k+m} (E^{+-1})^{ik} F_{kl}^+ (F^{+-st})^{lm} (E^{+st})_{mj}$ , hence we find  $\Pi^{ij}(g) = 0$ , i.e., for this subcase we must have super non-Abelian  $T$ -duality.

**Case 2:** In this case, the corresponding gluing supermatrix is given by

$$\mathcal{R}^i_j = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (5.33)$$

This is a  $D_{(-1)|2}$ -brane, with two Dirichlet directions for the bosonic part and two Neumann directions for the fermionic one. The dual gluing supermatrix again follows from (5.26):

$${}_i\tilde{\mathcal{R}}^j = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{e^{\tilde{x}}-2} & \frac{e^{\tilde{x}}}{e^{\tilde{x}}-2} & 0 \\ 0 & \frac{2\tilde{\psi}e^{\tilde{x}}(e^{\tilde{x}}-1)}{e^{\tilde{x}}-2} & \frac{e^{\tilde{x}}}{e^{\tilde{x}}-2} & 0 \\ 0 & 2\tilde{\chi}(e^{-\tilde{x}}-1) & 0 & 1-2e^{-\tilde{x}} \end{pmatrix}. \quad (5.34)$$

The superdeterminant is  $s\det({}_i\tilde{\mathcal{R}}^j) = 1$ . In this case, the dual branes may include the same directions of the dual branes in Case 1.

**Case 3:** In this case, we have a  $D_{0|0}$ -brane, with the following gluing supermatrix

$$\mathcal{R}^i_j = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad (5.35)$$

with one Dirichlet direction and one Neumann direction for the bosonic part and two Dirichlet directions for the fermionic one. Then Eq. (5.26) yields the dual gluing supermatrix

$${}_i\tilde{\mathcal{R}}^j = \begin{pmatrix} 1 & \frac{2\tilde{\psi}}{e^{\tilde{x}}-2}(\psi e^{\tilde{x}+y} + 2\tilde{\chi}) & \frac{-2(\tilde{\chi}+\psi e^y)}{e^{\tilde{x}}-2} & 2\tilde{\psi} \\ 0 & -1 & 0 & 0 \\ 0 & \frac{-2\tilde{\psi}e^{\tilde{x}}(e^{\tilde{x}}-1)}{e^{\tilde{x}}-2} & \frac{-e^{\tilde{x}}}{e^{\tilde{x}}-2} & 0 \\ 0 & 2\tilde{\chi}(1-e^{-\tilde{x}}) & 0 & 2e^{-\tilde{x}}-1 \end{pmatrix}. \quad (5.36)$$

The superdeterminant is  $s\det({}_i\tilde{\mathcal{R}}^j) = -1$ , so it is either a  $D_{0|0}$ -brane or a  $D_{0|2}$ -brane.

**Case 4:** This case introduces the following gluing supermatrix

$$\mathcal{R}^i_j = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (5.37)$$

This is a  $D_{0|2}$ -brane, with one Dirichlet direction and one Neumann direction for the bosonic part and zero Dirichlet directions for the fermionic one. The dual gluing supermatrix reads

$${}_i\tilde{\mathcal{R}}^j = \begin{pmatrix} 1 & 2\tilde{\psi}\psi e^{\tilde{x}+y} & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & \frac{2\tilde{\psi}e^{\tilde{x}}}{e^{\tilde{x}}-2} & \frac{e^{\tilde{x}}}{e^{\tilde{x}}-2} & 0 \\ 0 & -\tilde{A}_2 & 0 & 1 - 2e^{-\tilde{x}} \end{pmatrix}. \quad (5.38)$$

Its superdeterminant is  $-1$ , so, the dual branes are  $D_{0|0}$ -brane or  $D_{0|2}$ -brane.

**Case 5:** The corresponding gluing supermatrix for this case has the following form

$$\mathcal{R}^i_j = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (5.39)$$

This is a  $D_{1|0}$ -brane, with zero Dirichlet direction for the bosonic part and two Dirichlet directions for the fermionic one. The dual gluing supermatrix becomes

$${}_i\tilde{\mathcal{R}}^j = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & \frac{-2\tilde{\psi}e^{\tilde{x}}(e^{\tilde{x}}-1)}{e^{\tilde{x}}-2} & \frac{-e^{\tilde{x}}}{e^{\tilde{x}}-2} & 0 \\ 0 & 2\tilde{\chi}(1 - e^{-\tilde{x}}) & 0 & 2e^{-\tilde{x}} - 1 \end{pmatrix}. \quad (5.40)$$

It has superdeterminant  $s\det({}_i\tilde{\mathcal{R}}^j) = 1$ , so the dual branes may include the same directions of the dual branes in Case 1.

**Case 6:** This case is devoted to a spacefilling  $D$ -brane, i.e.,  $D_{1|2}$ -brane. The corresponding gluing supermatrix, according to Eq. (5.15) is given by

$$\mathcal{R}^i_j = (-1)^k (F^{+-1})^{ik} (F^{+st})_{kj} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2\psi e^y \\ 2\psi e^y & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (5.41)$$

Then, the dual gluing supermatrix is found to be of the form

$${}_i\tilde{\mathcal{R}}^j = \begin{pmatrix} -1 & \frac{-4\tilde{\psi}\tilde{\chi}}{e^{\tilde{x}}-2} & \frac{2\tilde{\chi}}{e^{\tilde{x}}-2} & -2\tilde{\psi} \\ 0 & -1 & 0 & 0 \\ 0 & \frac{2\tilde{\psi}e^{\tilde{x}}}{e^{\tilde{x}}-2} & \frac{e^{\tilde{x}}}{e^{\tilde{x}}-2} & 0 \\ 0 & 2e^{-\tilde{x}}\tilde{\chi} & 0 & 1 - 2e^{-\tilde{x}} \end{pmatrix}. \quad (5.42)$$



Its superdeterminant is 1, so for the dual branes we have the same directions of the dual branes in Case 1.

In this example, we showed how  $D$ -branes in the model are exchanged. At the end, we shall discuss the boundary conditions for Case 2. First, by insertion of relations (4.2) into (2.13), we compute the  ${}_{\mathcal{R}}R^{(l)i}$ . Then, by substituting (5.33) into the first equation of (5.17), we obtain

$$\mathcal{R}^{\mathcal{Y}}_{\Lambda} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 2e^y\chi & 0 \\ 0 & 0 & 1 & 0 \\ 2\chi & 0 & 0 & 1 \end{pmatrix}. \quad (5.43)$$

In terms of the  $\mathcal{R}^{\mathcal{Y}}_{\Lambda}$ , the gluing condition (5.2) implies the Dirichlet boundary condition  $\partial_{\tau}y = 0$  for the field  $y$ . The boundary conditions for the remaining fields are of the form

$$\partial_{\tau}x = e^y\chi \partial_{\tau}\psi, \quad \partial_{\sigma}\chi = -\chi \partial_{\sigma}y, \quad \partial_{\sigma}\psi = 0. \quad (5.44)$$

By imposing the latter condition on the first two equations, we then obtain  $\partial_{\tau}\partial_{\sigma}x = 0$ . i.e.,  $\partial_{\tau}x = f(\tau)$  or  $\partial_{\sigma}x = g(\sigma)$ . Similarly, using the (4.8) and (5.34),  $\tilde{\mathcal{R}}^{\mathcal{Y}}_{\Lambda}$  reads

$$\tilde{\mathcal{R}}^{\mathcal{Y}}_{\Lambda} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\frac{2\tilde{\psi}(e^{\tilde{x}}-1)}{e^{\tilde{x}}-2} & \frac{e^{\tilde{x}}}{e^{\tilde{x}}-2} & 0 \\ 0 & 2\tilde{\chi}(1-e^{-\tilde{x}}) & 0 & 1-2e^{-\tilde{x}} \end{pmatrix}. \quad (5.45)$$

Now, by employing (5.2), the dual boundary conditions are found to be

$$\begin{aligned} \partial_{\sigma}\tilde{y} &= 0, \\ \partial_{\sigma}\tilde{x} &= 0, \\ -\tilde{\psi}(e^{\tilde{x}}-1)\partial_{\tau}\tilde{x} + (e^{\tilde{x}}-1)\partial_{\sigma}\tilde{\psi} + \partial_{\tau}\tilde{\psi} &= 0, \\ \tilde{\chi}(e^{\tilde{x}}-1)\partial_{\tau}\tilde{x} + (e^{\tilde{x}}-1)\partial_{\sigma}\tilde{\chi} - \partial_{\tau}\tilde{\chi} &= 0. \end{aligned} \quad (5.46)$$

Note that the latter two equations imply the following condition

$$(e^{\tilde{x}}-1)\partial_{\sigma}(\tilde{\chi}\tilde{\psi}) + \tilde{\chi}\partial_{\tau}\tilde{\psi} + \tilde{\psi}\partial_{\tau}\tilde{\chi} = 0. \quad (5.47)$$

## 6 Conclusion

We have proved that the WZNW model on the Lie supergroup  $GL(1|1)$  has super Poisson-Lie symmetry with the dual Lie supergroup  $B \oplus A \oplus A_{1,1|i}$ . Then, we discussed about  $D$ -branes and worldsheet boundary conditions on supermanifolds, in general, and obtained the algebraic relations on the gluing supermatrix for the Lie supergroup case. Also, using the supercanonical transformation description of the super Poisson-Lie T-duality transformation, we obtained formulae for the description of the dual gluing supermatrix, then, we found the gluing supermatrix for the WZNW model on  $GL(1|1)$  and its dual model. In this way, there are some new perspectives to find super Poisson-Lie

symmetry on the superstring models on Ads backgrounds. Investigation of superconformal boundary conditions [27] on supermanifolds may become another open problem. Some of these open problems are under current investigation.

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